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HARMONIC GENERATION AND ANISOCHRONISM IN MAGNETOSTRICTIVE MATERIALS

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Abstract—The present work uses the nonlinear, rotationally invariant equations of the magnetoelasticity of anisotropic magnetostrictive materials to provide the basic elements of nonlinear bulk wave propagation in these materials. In particular, near- and far-field solutions (the latter being uniformly valid on long spatial intervals) from a harmonic source within the framework of the monomode hypothesis are given using either a straightforward expansion of the magnetoacoustic solution in small parameters or a more refined multiple-scale technique. A bias magnetic field is necessarily present and harmonics are generated through all nonlinear features with a special attention to magnetostrictive couplings. A closed-form expression is deduced for the far-field solution at the first order in the small parameters. In the case of an elastic resonator tuned on the first partial mode and placed in a bias magnetic field, the expansion method provides the anisochronism due to magnetostrictive couplings at the second order. Anisochronism caused by the nonlinear purely elastic behavior requires solving the hierarchy of approximate boundary-value problems to higher order. In all, the work presents all the prerequisites for a forthcoming study of nonlinear surface magnetoelastic waves and a more complete study of nonlinear vibrations of magnetoelastic resonators.

1. INTRODUCTION

In a previous paper (Abd-Alla and Maugin, 1987) we have deduced sets of nonlinear partial differential equations and accompanying boundary conditions that govern nonlinear magnetoacoustic problems in the bulk and at a surface and that include terms up to the third order jointly in the gradient of the displacement and the gradient of the quasi-magnetostatic potential. This allows one to envisage nonlinearities of pure mechanical, pure magnetic and mixed magnetoelastic origins.

It is well known that the propagation characteristics of both bulk and surface waves in centrosymmetric magnetostrictive materials (such as ferromagnetic polycrystalline materials) can be utilized to build a number of signal-processing devices such as electro-magneto-acoustic transducers (so-called EMATs, Hauser *et al.*, 1981; Ristic, 1983; Thompson, 1981; Worley, 1971). The *nonlinear* magnetoelastic couplings have not received much attention although "linearized" magnetostriction in the presence of a bias magnetic field has been considered in both ferromagnets (Maugin, 1979a) and paramagnets (Maugin and Hakmi, 1984) with a view to studying magnon–phonon couplings and small-amplitude wave propagation, and magnetostriction is one of the coupling mechanisms which afford the conception of delay lines and transducers. However, contrary to piezoelectricity (which is rather common but of a varying strength depending on the material) and piezomagnetism (which is rare), electrostriction in electrically polarizable bodies and magnetostriction in magnetizable materials are *nonlinear* coupling phenomena. They have associated with them a stress, or an internal strain (see e.g. Maugin, 1979b), that does not depend on the direction of the applied field (an electric field in the former case, a magnetic field in the latter) so that it is of even order (e.g. at least quadratic), and not ruled out by restrictive symmetry regulations, in the said field. From a rather different point of view, recent works (Maugin, 1985, 1988; Nelson, 1978, 1979; Planat, 1984) have developed to some extent the field of nonlinear electromagneto-mechanical wave propagation.

In the present work, with a view to studying the main nonlinear wave characteristics of magnetomechanical signal processing devices (for example, resonators), we exhibit useful

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solutions of some nonlinear problems by using approximation methods (such as in Nayfeh, 1973; Nayfeh and Mook, 1979; Whitham, 1974) such as the straightforward expansion in a small parameter and the multiple-scale technique. Regarding the main results of this work, one finds, as for corresponding electromechanical problems, that these two methods essentially yield equivalent results for a short distance of propagation, while the multiple-scale technique is to be used if one wants to obtain a solution that is uniformly valid on a long spatial interval. Theoretically, we have found that the nonlinearities obviously generate harmonics in the magnetoacoustic field while in the case of resonators the phenomenon of anisochronism (relative change in velocity of the fundamental frequency component) is placed in evidence. These phenomena are very similar to those obtained in electromechanical devices (Planat, 1984; Maugin, 1985). However, there are also fundamental differences that come from three facts: (i) the material body being centrosymmetric, the first existing magnetoelastic coupling is of higher order than piezoelectricity for electromechanical devices. This brings into the picture the second fact (ii) that the lowest order coupled linearized solution must necessarily involve a *bias* magnetic field. Finally, (iii) there does not exist for magnetic processes the equivalent of grounding (i.e. imposing a potential) at a surface and this, of necessity, yields a matching of an internal solution for the magnetic field to an external solution as soon as limiting (boundary) surfaces are involved. This is the case for the "resonator" configuration and this will also create some difficulties in a forthcoming study of linear and nonlinear magnetoelastic surface waves.

The needed nonlinear magnetoacoustic equations derived in a previous paper (Abad-Alla and Maugin, 1987) are given in Section 2. The linear wave equations which provide a natural basis (related to eigenmodes) to study all subsequent nonlinear and coupling phenomena are dealt with in Section 3. More interesting for our purpose are the equations obtained by linearization about a bias magnetic field (Section 4). The nonlinear equations, but for a monomode process, are given in Section 5. These equations that contain all types of nonlinearities, are first exploited for bulk waves in Section 6 by using a straightforward expansion in small parameters. As in other fields of mathematical physics [nonlinear (fluid) acoustics, nonlinear elasticity, nonlinear electromechanical processes], this yields only a near-field solution and this limitation is remedied in Section 7 by looking for a uniformly valid far-field solution via a multiple-scale technique. The solution obtained is close to the celebrated Fubini–Ghiron solution but internal, spatially uniform stresses result from the bias magnetic field involved in the zeroth-order solution. The approximation of classical magnetostriction (i.e. magnetostriction is regarded as the only nonlinear process, nonlinear elastic and purely magnetic phenomena being discarded) is dealt with along the same lines in Section 8. It is shown in Section 9 that taking account of nonlinearities of all origins then brings only an alteration in the coefficients of the previous solution. Finally, the case of elastic magnetostrictive resonators is examined in Section 10 by using a straightforward expansion method. There is exhibited a defect called anisochronism which is directly proportional to the square of the bias magnetic field and the magnetoacoustic coupling coefficient.

2. EQUATIONS OF NONLINEAR MAGNETOELASTICITY

The equations of motion and the equations of the quasi-stationary magnetic field (i.e. in the framework of quasi-magnetostatics for acoustic frequencies) for material points **X** inside a regular body occupying the region D_n of three-dimensional Euclidean space in the reference configuration K_R of continuum mechanics may be written in the following material form (Abd-Alla and Maugin, 1987).

$$\rho_R \frac{\partial^2 u_i}{\partial t^2} = (\tilde{T}^E_{Ki} + T^E_{Ki})_{,K} \tag{1}$$

and

$$\mathfrak{B}_{K,K} = 0, \quad \mathfrak{H}_K = -\phi_{,K} \tag{2a, b}$$

where we have used the convention that upper- and lower-case Latin indices refer, respectively, to the reference Cartesian coordinates and to the present Cartesian coordinates of material points $\mathbf{X}$. We also employ the convention that a comma followed by a (capital) index denotes partial differentiation with respect to the reference coordinates, and repeated indices are to be summed in agreement with Einstein's convention. The symbols ρ_R, $u_i = \mathscr{X}_i - X_K\delta_{Ki}$, $\mathscr{X}_i$, $\mathfrak{B}_K$, $\mathfrak{H}_K$, ϕ and $T'_{Ki} = \tilde{T}^E_{Ki} + T^F_{Ki}$ denote, respectively, the matter density at the reference state, the elastic displacement, the nonlinear motion, the "Lagrangian" magnetic induction, the "Lagrangian" magnetic field, the magnetostatic scalar potential and the total Piola–Kirchhoff (matter contribution plus field contribution) nonsymmetric stress. The nonlinear stress constitutive equation in the material obviously considered as an insulator is obtained for T'_{Ki} on a thermodynamical basis as (Abd-Alla and Maugin, 1987)

$$T'_{Ki} = \delta_{Ri}\{C_{KRMN}u_{M,N} + \gamma'_{KRMNPQ}u_{M,N}u_{P,Q} + \delta'_{KRMNPQAB}u_{M,N}u_{P,Q}u_{A,B} \\ + B'''_{KRMN}\mathfrak{H}_M\mathfrak{H}_N + B'''_{KRMNPQ}\mathfrak{H}_M\mathfrak{H}_N u_{P,Q}\} \tag{3}$$

while one can show that eqn (2a) takes on the form

$$\mu_{KL}\mathfrak{H}_{L,K} - B''_{KLMN}(\mathfrak{H}_L u_{M,N})_{,K} + \tfrac{1}{6}\chi_{KLMN}(\mathfrak{H}_L\mathfrak{H}_M\mathfrak{H}_N)_{,K} - \lambda''_{KLMNPQ}(\mathfrak{H}_L u_{M,N}u_{P,Q})_{,K} = 0 \tag{4}$$

where we have set

$$\gamma'_{KRMNPQ} = \tfrac{1}{2}(C_{KRMNPQ} + C_{KRPN}\delta_{QM}) + C_{KQMN}\delta_{RP} \tag{5a}$$

$$\delta'_{KRMNPQAB} \doteq \tfrac{1}{2}\{\tfrac{1}{3}C_{KRMNPQAB} + \tfrac{1}{2}C_{KRMNAQ}\delta_{BP} \\ + \tfrac{1}{2}C_{KRANPQ}\delta_{BM} + C_{KRMNPQ}\delta_{AB} + C_{KRPN}\delta_{QM}\delta_{AB}\} \tag{5b}$$

$$B'''_{KRMN} = \tfrac{1}{2}B_{KRMN} + \chi_{RN}\chi_{KM} + \delta_{KM}\delta_{NR} - \tfrac{1}{2}\delta_{NM}\delta_{KR} \tag{5c}$$

$$B'''_{KRMNPQ} = B_{KRMNPQ} + \tfrac{1}{2}B_{KQMN}\delta_{RP} - B_{MNPQ}\chi_{KR} - B_{RKPQ}\chi_{MN} - \chi_{RN}\chi_{KP}\delta_{QM} - \chi_{RN}\chi_{KQ}\delta_{MP} \\ + \chi_{PM}\chi_{KN}\delta_{QR} + \chi_{KM}\chi_{RN}\delta_{PQ} + \delta_{NP}(\tfrac{1}{2}\delta_{QM}\delta_{KR} - \delta_{KM}\delta_{QR}) + \delta_{KP}(\tfrac{1}{2}\delta_{NM}\delta_{QR} - \delta_{QM}\delta_{NR}) \\ + \delta_{MP}(\tfrac{1}{2}\delta_{QN}\delta_{KR} - \delta_{KN}\delta_{QR}) + \delta_{PQ}(\delta_{KM}\delta_{NR} - \tfrac{1}{2}\delta_{MN}\delta_{KR}) \tag{5d}$$

$$B''_{KLMN} = B_{KLMN} + \delta_{LM}\delta_{NK} + \delta_{KM}\delta_{NL} + \delta_{KL}\delta_{NM} \tag{5e}$$

$$\lambda''_{KLMNPQ} = \lambda_{KLMNPQ} + \delta_{QL}\delta_{KM}\delta_{NP} - \delta_{NQ}(\delta_{LP}\delta_{KM} + \tfrac{1}{2}\delta_{PM}\delta_{KL}) \\ + \delta_{NM}(\delta_{KP}\delta_{QL} - \tfrac{1}{2}\delta_{PQ}\delta_{KL}) - \delta_{QK}(\delta_{LM}\delta_{NP} - \delta_{LP}\delta_{NM}) \tag{5f}$$

$$\mu_{KL} = \delta_{KL} + \chi_{KL}. \tag{5g}$$

The various material coefficients introduced have obvious tensorial symmetries and they bear the following significance. The material tensors C_{KLMN}, C_{KLMNPQ} and $C_{KRMNPQAB}$ are the tensors of elasticity coefficients of the second, third and fourth orders, at constant temperature and zero magnetic field. The material tensors χ_{AB} and χ_{KRMN} are tensors of magnetic susceptibilities of the second and fourth orders at constant temperature and vanishing strains. The material tensors B_{MNKL} and B_{MNKLPQ} are the tensors of coefficient· of magnetostriction of the first and second orders (in the strain), respectively. Finally, th symbol δ_{Ri} is a translation operator (shifter) from the reference configuration K_R to tl actual configuration K_t and vice versa.

At a fixed regular (material) boundary ∂D_o of the body equipped with unit outward normal of components N_K, the field equations (1) and (2) are complemented by the following boundary (jump) conditions in the absence of prescribed tractions and surface currents:

$$N_K[\![T'_{Ki}]\!] = 0 \tag{6a}$$

$$N_K[\![\mathfrak{B}_K]\!] = 0 \tag{6b}$$

$$[\![\phi]\!] = 0 \tag{6c}$$

where $[\![..]\!]$ denotes the jump (difference between the outside and inside values) of the enclosure. We note that

$$\mathfrak{B}_K = \mu_{KL}\mathfrak{H}_L - B''_{KLMN}\mathfrak{H}_L u_{M,N} + \tfrac{1}{6}\chi_{KLMN}\mathfrak{H}_L\mathfrak{H}_M\mathfrak{H}_N - \lambda''_{KLMNPQ}\mathfrak{H}_L u_{M,N}u_{P,Q}.$$

3. LINEAR EQUATIONS

The part of eqn (1) which is linear in the *two* field variables is

$$\rho_R\ddot{u}_R - C_{KRMN}u_{M,NK} = 0 \tag{7}$$

where a superimposed dot denotes partial time differentiation. This is an equation of motion for a linear anisotropic elastic body (crystal). A solution of eqn (7) may be sought in the form of a plane wave travelling in the direction of unit (material) vector $\boldsymbol{\lambda}$ with velocity V_o. We write

$$u_M = b_M \exp\left[i(\omega t - \mathbf{K}\cdot\mathbf{X})\right] \tag{8}$$

where $\mathbf{K} = k\boldsymbol{\lambda}$; k, real, represents the wave number, ω is the angular frequency and $\mathbf{b}$ is the amplitude. Carrying (8) into (7) we obtain

$$(C_{KRMN}\lambda_N\lambda_K - \rho_R V_o^2\delta_{MR})b_M = 0. \tag{9}$$

This is the usual eigenvalue problem for the acoustic Christoffel tensor $\Gamma_{RM}(\boldsymbol{\lambda})$ defined by

$$\Gamma_{RM}(\boldsymbol{\lambda}) = C_{KRMN}\lambda_N\lambda_K = \Gamma_{MR}(\boldsymbol{\lambda}) = \Gamma_{RM}(-\boldsymbol{\lambda}). \tag{10}$$

Let $m_\alpha = \rho_R(V_o^\alpha)^2$, $\alpha = 1,2,3$, be the three eigenvalues of Γ_{RM}. The fact that $C_{KRMN} \Leftrightarrow C_{\alpha\beta}$ (in Voigt's notation with $\alpha, \beta = 1, 2, \ldots, 6$) is symmetric positive definite guarantees that the three eigenvalues are positive and, in general, distinct from each other. The polarizations $\mathbf{b}$ of the associated vibrations are orthogonal to one another. Because of the general degree of anisotropy these vibrations, in general, do not correspond to purely transverse or purely longitudinal (with respect to $\boldsymbol{\lambda}$) vibrations. The eigenvalues m_α are obtained by solving

$$\det|\Gamma_{RM}(\boldsymbol{\lambda}) - m\,\delta_{RM}| = 0 \tag{11}$$

which is a cubic in m for a fixed $\boldsymbol{\lambda}$. The general solution of (7) is a linear combination of the three elementary eigenmodes $\mathbf{u}^\alpha$, i.e.

$$u_M = \sum_{\alpha=1}^{3} A_\alpha u_M^\alpha \tag{12}$$

where the A_α are arbitrary amplitudes. Let

$$l_R^\alpha = \frac{b_R^\alpha}{\sqrt{b_K^\alpha b_K^\alpha}}; \quad \alpha \text{ fixed} \tag{13}$$

denote the director cosines of the eigenmodes. The $\{\mathbf{l}^\alpha; \alpha = 1, 2, 3\}$ form a new triad of orthonormal vectors, hence a Cartesian basis on which a general motion can be described. In particular, each mode then is characterized by a scalar displacement u^α which can be expressed in terms of time and the curvilinear abscissa $\mathfrak{X} = \boldsymbol{\lambda}\cdot\mathbf{X} = \lambda_K X_K$. We have

$$u^\alpha = A_\alpha u_M^\alpha l_M^\alpha \quad \text{(no summation on } \alpha\text{)}. \tag{14}$$

Substituting from this into eqn (7) we get

$$\rho_R \sum_\alpha \ddot{u}^\alpha l_R^\alpha - C_{KRMN} \sum_\alpha \lambda_N \lambda_K l_M^\alpha u^\alpha_{,\mathfrak{X}\mathfrak{X}} = 0. \tag{15}$$

This vectorial equation (three components) can be projected onto the basis $\{\mathbf{l}^\alpha\}$ on account of the fact that

$$l_K^\alpha l_K^\beta = \delta^{\alpha\beta}, \quad \sum_\alpha l_K^\alpha l_L^\alpha = \delta_{KL}. \tag{16}$$

By virtue of the very definition of an eigenmode, one obtains thus

$$\rho_R \ddot{u}^\alpha = m_\alpha u^\alpha_{,\mathfrak{X}\mathfrak{X}} \tag{17}$$

with

$$m_\alpha = C_{KRMN} \lambda_K \lambda_M l_N^\alpha l_R^\alpha = l_M^\alpha \Gamma_{MR} l_R^\alpha. \tag{18}$$

4. THE LINEARIZED EQUATIONS

When one studies the propagation of small disturbances in elastic displacement $\tilde{\mathbf{u}}$ and magnetic field $\tilde{\mathfrak{H}}$ (with associated potential $\tilde{\phi}$) superimposed on a state of spatially uniform magnetic field $\mathbf{H}^o$ and zero displacement, by linearization of eqns (1) and (2) one obtains the following linear equations which resemble those of linear piezoelectricity but for the fact that the coupling tensorial coefficients F'_{KLN} and F^*_{KLN} are *not* the same ones:

$$\rho_R \ddot{\tilde{u}}_L - C_{KLMN} \tilde{u}_{M,NK} + F'_{KLN} \tilde{\phi}_{,NK} = 0 \tag{19}$$

$$\mu_{KL} \tilde{\phi}_{,LK} + F^*_{KMN} \tilde{u}_{M,NK} = 0 \tag{20}$$

where (here the bias field $\mathbf{H}^o$ may have any direction)

$$F^*_{KMN} = B^o_{KLMN} H^o_L, \quad H^o_L = -\phi^o_{,L} \tag{21}$$

$$F'_{KLN} = 2B_{KLMN} H^o_M. \tag{22}$$

By projecting eqns (19) and (20) onto the basis $\{\mathbf{l}^\alpha\}$ of Section 3 we obtain

$$\rho_R \ddot{\tilde{u}}^\alpha = m_\alpha \tilde{u}^\alpha_{,\mathfrak{X}\mathfrak{X}} - \bar{F}'^\alpha \tilde{\phi}_{,\mathfrak{X}\mathfrak{X}} \tag{23}$$

$$\bar{\mu} \tilde{\phi}_{,\mathfrak{X}\mathfrak{X}} + \sum_\alpha \bar{F}^\alpha \tilde{u}^\alpha_{,\mathfrak{X}\mathfrak{X}} = 0 \tag{24}$$

where

$$\bar{F}' = \sum_{\alpha} F'_{KLN}\lambda_K\lambda_N l^{\alpha}_L \tag{25}$$

$$\bar{F}^{\alpha} = F^{*}_{KMN}\lambda_N\lambda_K l^{\alpha}_M, \quad \bar{\mu} = \mu_{KL}\lambda_K\lambda_L. \tag{26}$$

We can write eqns (23) and (24) in the following form (for a fixed α) by considering the following scaling (L and T are characteristic length and time scales),

$$\left.\begin{aligned} u &= Lu^*, \quad \mathfrak{X} = L\mathfrak{X}^* \\ t &= Tt^*, \quad H = H''H^*, \quad H'' = |\mathbf{H}''| \\ \phi_{,\mathfrak{X}} &= H''\phi^*_{,\mathfrak{X}}, \end{aligned}\right\} \tag{27}$$

so that the above equations become dimensionless. We have thus (omitting the asterisks and with the simplified notation $u_t = \partial u/\partial t$, $u_x = \partial u/\partial x$, $x = \mathfrak{X}$),

$$u_{tt} - u_{xx} = -\beta_1\phi_{xx} \tag{28}$$

$$\phi_{xx} + \beta_2 u_{xx} = 0 \tag{29}$$

where we have set

$$\beta_1 = B(H'')^2/m, \quad \beta_2 = \bar{B}''/\bar{\mu}. \tag{30}$$

Substituting from (29) into (28) yields the unique equation

$$u_{tt} - K^{-2}u_{xx} = 0 \tag{31}$$

with

$$K^2 = \frac{1}{1+\varepsilon_m} \simeq 1 - \varepsilon_M, \quad \varepsilon_m \equiv \beta_1\beta_2 \tag{32}$$

where ε_m may be called the magnetoacoustic coupling coefficient (it describes the *reduction* in elastic wave speed as a result of magnetostrictive couplings—see Maugin and Hakmi, 1984).

With a boundary condition $u(x = 0, t) = U_0(1-\cos\omega t)$ at the source, eqn (31) has the obvious solution

$$u = U_0(1-\cos\psi), \quad \psi = \omega t - k_0 x, \quad k_0 = \omega/K. \tag{33}$$

5. NONLINEAR MONOMODE EQUATIONS

Projecting the nonlinear equations (1) and (4) onto the basis $\{\mathbf{l}^{\alpha}\}$ of Section 3 we obtain

$$\rho_R\ddot{u}^{\alpha} = m_{\alpha}u^{\alpha}_{,\mathfrak{X}\mathfrak{X}} + \sum_{\beta,\gamma}\Gamma_{\alpha\beta\gamma}(u^{\beta}_{,\mathfrak{X}}u^{\gamma}_{,\mathfrak{X}})_{,\mathfrak{X}} + \sum_{\beta,\gamma,\delta}\Delta_{\alpha\beta\gamma\delta}(u^{\beta}_{,\mathfrak{X}}u^{\gamma}_{,\mathfrak{X}}u^{\delta}_{,\mathfrak{X}})_{,\mathfrak{X}} + B_{\alpha}(\phi_{,\mathfrak{X}}\phi_{,\mathfrak{X}})_{,\mathfrak{X}} + \sum_{\delta}\bar{B}_{\alpha\delta}(\phi_{,\mathfrak{X}}\phi_{,\mathfrak{X}}u^{\delta}_{,\mathfrak{X}})_{,\mathfrak{X}} \tag{34}$$

and

$$\bar{\mu}\phi_{,\mathfrak{X}\mathfrak{X}} - \sum_{\beta}B''_{\beta}(\phi_{,\mathfrak{X}}u^{\beta}_{,\mathfrak{X}})_{,\mathfrak{X}} + \tfrac{1}{6}\bar{\chi}(\phi_{,\mathfrak{X}})^3_{,\mathfrak{X}} - \sum_{\beta,\gamma}\bar{\lambda}''_{\beta\gamma}(\phi_{,\mathfrak{X}}u^{\beta}_{,\mathfrak{X}}u^{\gamma}_{,\mathfrak{X}})_{,\mathfrak{X}} = 0 \tag{35}$$

where we have set

$$\left.\begin{aligned}
\Gamma_{\alpha\beta\gamma} &= \gamma'_{KRMNPQ}\lambda_N\lambda_Q\lambda_K l^{\alpha}_M l^{\beta}_P l^{\gamma}_R \\
\Delta_{\alpha\beta\gamma\delta} &= \delta'_{KRMNPQAB}\lambda_N\lambda_Q\lambda_B\lambda_K l^{\alpha}_M l^{\beta}_P l^{\delta}_A l^{\gamma}_R \\
B_{\alpha} &= B'''_{KRMN}\lambda_M\lambda_N\lambda_K l^{\alpha}_R \\
\bar{B}_{\alpha\beta} &= B'''_{KRMNPQ}\lambda_M\lambda_N\lambda_Q\lambda_K l^{\beta}_P l^{\alpha}_R \\
\bar{\mu} &= \mu_{KL}\lambda_K\lambda_L \\
\bar{B}''_{\beta} &= B''_{KLMN}\lambda_K\lambda_N\lambda_L l^{\beta}_M \\
\bar{\lambda}''_{\beta\gamma} &= \lambda''_{KLMNPQ}\lambda_L\lambda_N\lambda_Q\lambda_K l^{\beta}_M l^{\gamma}_P \\
\bar{\chi} &= \chi_{KLMN}\lambda_L\lambda_M\lambda_N\lambda_K.
\end{aligned}\right\} \tag{36}$$

Therefore, the nonlinearities cause a mutual coupling between eigenmodes (case where β and/or γ are different from α and β, γ and/or δ are different from α), as also a *self-coupling* of each mode (for β and/or $\gamma = \alpha$, and β, γ and/or $\delta = \alpha$). At this point the following remark is of importance since it will greatly simplify further considerations. The three eigenmodes of the linear case are orthogonal and propagate with different velocities. Therefore, when one specific mode is excited by the magnetic field (or a mechanical agent), and the other modes correspond to eigenfrequencies which are sufficiently remote from the excited one, the latter is preponderant and couplings with the other modes correspond to corrective terms which may be neglected in a first approach (cf. Sugimoto, 1978; see also Maugin, 1985, p. 36). This corresponds to the working hypothesis of a monomode process. This hypothesis is commonly considered in the case of electromechanical interactions both for bulk waves (e.g. Planat *et al.*, 1980) and surface waves (e.g. Kalyanasundaram, 1984). In this condition only the terms corresponding to a selected $\alpha = \beta = \gamma = \delta$ in eqns (34) and (35) are kept and, using the notation introduced for eqn (28), and omitting the superscript α, eqns (34) and (35) take on the following form,

$$u_{tt} - u_{xx}\left[\frac{m}{\rho_R}\left(1 + \frac{2\Gamma}{m}u_x + \frac{3\Delta}{m}u_x^2\right)\right] = \frac{2B}{\rho_R}\phi_x\phi_{xx} + \frac{\bar{B}}{\rho_R}(\phi_x^2 u_x)_x \tag{37}$$

and

$$\bar{\mu}\phi_{xx} - \bar{B}''(\phi_x u_x)_x + \tfrac{1}{6}\bar{\chi}(\phi_x)_x^3 - \bar{\lambda}''(\phi_x u_x^2)_x = 0. \tag{38}$$

Equations (37) and (38) can now be expressed in dimensionless form by using the same scaling as in eqns (27). Performing the nondimensionalization and omitting the asterisks to lighten the notation, we write eqns (37) and (38) as

$$u_{tt} - u_{xx}(1 + 2\gamma u_x + 3\delta u_x^2) = \beta_1(\phi_x)_x^2 + \beta_1(\phi_x^2 u_x)_x \tag{39}$$

and

$$\phi_{xx} - \beta_2(\phi_x u_x)_x + \tfrac{1}{6}\chi(\phi_x)_x^3 - \lambda(\phi_x u_x^2)_x = 0 \tag{40}$$

where we have set (orders of magnitude are mentioned)

$$\left.\begin{aligned}
\gamma &= \frac{\Gamma}{m} = 0(1) \\
\delta &= \frac{\Delta}{m} = 0(1 \text{ to } 10^{-1}) \\
\beta_1 &= \frac{B(H^0)^2}{m} = \frac{\bar{B}(H^0)^2}{m} = 0(10^{-5}) \\
\beta_2 &= \frac{\bar{B}''}{m} = 0(1) \\
\chi &= \frac{\bar{\chi}(H^0)^2}{\bar{\mu}} = 0(10^{-4}) \\
\lambda &= \frac{\bar{\lambda}''}{\bar{\mu}} = 0(1).
\end{aligned}\right\} \tag{41}$$

It is for these orders of magnitude that $\bar{B}'' = 0(10^{-5})$, $\bar{\mu} = 0(10^{-5})$ and $\bar{\chi} = 0(10^{-9})$.

For the sake of simplicity we shall discard the last term in each of eqns (39) and (40), obtaining thus

$$u_{tt}-u_{xx}(1+2\gamma u_x+3\delta u_x^2) = \beta_1(\phi_x^2)_x \tag{42}$$

$$\phi_{xx}-\beta_2(\phi_x u_x)_x+\tfrac{1}{6}\chi(\phi_x^3)_x = 0. \tag{43}$$

We assume the following boundary condition at the source (harmonic source with a single frequency normalized to one).

$$u(x=0,t) = U_0(1-\cos t) \tag{44}$$

while, initially (i.e. $\mathbf{H}''$ is set along the x-axis or has an intense non-zero x-component at least)

$$\phi_x(x,0) = -H'' = \text{const.}\ \forall x\ (\text{i.e. } \nabla H'' = \mathbf{0}). \tag{45}$$

6. SOLUTIONS BY MEANS OF A STRAIGHTFORWARD EXPANSION

In accordance with the methodology of the perturbation method of Poincaré (straightforward expansion in a small parameter), we assume that the displacement component u and the gradient of the potential field ϕ_x depend on the space–time propagation variables x and t via an expansion in power series, for instance

$$u(x,t) = \epsilon_1 u^{(0)}(x,t)+\epsilon_1^2 u^{(1)}(x,t)+\epsilon_1^3 u^{(2)}(x,t)+\cdots \tag{46}$$

and

$$\phi_x(x,t) = -H^{(0)}+\epsilon_2\phi_x^{(1)}(x,t)+\epsilon_2^2\phi_x^{(2)}(x,t)+\cdots \tag{47}$$

where the small parameters ϵ_1 and ϵ_2 are introduced on account of (41) as (classically $u_x = 0(10^{-4})$)

$$\left.\begin{aligned} \epsilon_1 &\simeq \gamma u_x \simeq \beta_1 = 0(10^{-4} \text{ to } 10^{-5})\\ \epsilon_2 &\simeq \beta_2 u_x \simeq \chi = 0(10^{-4}). \end{aligned}\right\} \tag{48}$$

That is, ϵ_1 and ϵ_2 are small parameters of the same order, i.e. $\epsilon_1 = 0(\epsilon_2)$.

We substitute from eqns (46) and (47) into eqns (42) and (43) and set separately equal to zero the coefficients of various powers of ϵ_1 and ϵ_2, obtaining thus the following hierarchy of coupled one-dimensional magnetoacoustic problems.

- *Order one in ϵ_1 and ϵ_2*

$$u_{tt}^{(0)}-u_{xx}^{(0)} = 0 \tag{49}$$

$$\phi_{xx}^{(1)}+\beta_2 H^{(0)}u_{xx}^{(0)} = 0. \tag{50}$$

On account of the boundary condition (44), eqn (49) integrates at once to

$$u^{(0)} = U_0(1-\cos\Psi),\quad \Psi = t-x. \tag{51}$$

Substituting now from (51) into (50) one gets

$$\phi_{xx}^{(1)} = -\beta_2 H^{(0)} U_0 \cos \Psi. \tag{52}$$

With $\phi_x^{(1)} = 0$ at $x = 0$, $t = 0$, since $\phi_x^{(0)} = -H^{(0)}$, we obtain

$$\phi_x^{(1)} = \beta_2 H^{(0)} U_0 \sin \Psi. \tag{53}$$

- *Order two in ϵ_1 and ϵ_2*

  We have the equations

$$u_{tt}^{(1)} - u_{xx}^{(1)} = \gamma(u_x^{(0)^2})_x - 2\beta_1 H^{(0)} \phi_{xx}^{(1)} \tag{54}$$

and

$$\phi_{xx}^{(2)} - \beta_2[(\phi_x^{(1)} u_x^{(0)})_x - H^{(0)} u_{xx}^{(1)}] + \tfrac{1}{2}\chi H^{(0)^2} \phi_{xx}^{(1)} = 0. \tag{55}$$

Carrying (50) into (54) one has an equation involving only elastic displacements:

$$u_{tt}^{(1)} - u_{xx}^{(1)} = \gamma(u_x^{(0)})_x^2 + 2\varepsilon_m u_{xx}^{(0)}, \quad \varepsilon_m = \bar{\beta}_1 \bar{\beta}_2 = \beta_1 \beta_2 H^{(0)^2}. \tag{56}$$

One seeks a solution $u^{(1)}$ of this equation in the form (accounting for (51) in the right-hand side)

$$u^{(1)}(x, t) = A(x) \cos 2\Psi + B(x) \sin \Psi + C(x) \tag{57}$$

and, by using the method of "variation of constants", under the condition that

$$A'(x) \cos 2\Psi + B'(x) \sin \Psi + C'(x) = 0$$

we can find the spatially dependent coefficients as

$$\left.\begin{aligned} A(x) &= \gamma \frac{U_0^2}{2} x \\ B(x) &= 2\varepsilon_m U_0 x \\ C(x) &= -\gamma \frac{U_0^2}{2} x - 2\varepsilon_m U_0. \end{aligned}\right\} \tag{58}$$

Then after some calculations which amount to substituting from (57) and (58) into eqn (55) we obtain $\phi_{xx}^{(2)}$ as

$$\phi_{xx}^{(2)} = \beta_2 H^{(0)} U_0 \{2\gamma U_0 x \cos 2\Psi + (\beta_2 - \gamma) U_0 \sin 2\Psi \\ - 2\varepsilon_m x \sin \Psi + (\tfrac{1}{2}\chi - 2\beta_1 \beta_2) H^{(0)^2} \cos \Psi\}. \tag{59}$$

This can be integrated once in space with $\phi_x^{(2)}(0,0) = 0$ to obtain $\phi_x^{(2)}$. However, we need only (59) to proceed to the next order for the elastic displacement. $\phi_x^{(2)}$ is obtained as

$$\phi_x^{(2)} = -\beta_2 H^{(0)} U_0 \{\gamma U_0 x \sin 2\Psi - 2\varepsilon_m x \cos \Psi - \tfrac{1}{2}\beta_2 U_0 \cos 2\Psi - \tfrac{1}{2}\chi H^{(0)^2} \sin \Psi + \tfrac{1}{2}\beta_2 U_0\}.$$

- *Order three in ϵ_1 and ϵ_2*

  We have the equations

$$u_{tt}^{(2)} - u_{xx}^{(2)} = 2\gamma(u_x^{(0)} u_x^{(1)})_x + \delta(u_x^{(0)^3})_x + \beta_1[(\phi_x^{(1)})_x^2 - 2H^{(0)} \phi_{xx}^{(2)}] \tag{60}$$

and

$$\phi^{(3)}_{xx} - \beta_x[(\phi^{(1)}_x u^{(1)}_x)_x + (\phi^{(2)}_x u^{(0)}_x)_x - H^{(0)} u^{(2)}_{xx}] + \tfrac{1}{2}\chi[H^{(0)^2}\phi^{(2)}_{xx} - H^{(0)}(\phi^{(1)^2}_x)_x] = 0. \quad (61)$$

The expressions of $u^{(0)}$, $u^{(1)}$, $\phi^{(1)}_x$ and $\phi^{(2)}_{xx}$ suggest to seek $u^{(2)}$ in the form

$$u^{(2)} = D(x)\cos 3\Psi + E(x)\sin 3\Psi + F(x)\cos 2\Psi + G(x)\sin 2\Psi + H(x)\cos\Psi + I(x)\sin\Psi \quad (62)$$

and varying the "constants" allows one to find the spatially varying coefficients as

$$\left.\begin{aligned} D(x) &= -\tfrac{1}{2}\gamma^2 U_0^3 x^2 \\ E(x) &= \tfrac{1}{3}U_0^3(\gamma^2 - \tfrac{3}{4}\delta)x \\ F(x) &= \tfrac{1}{2}U_0[\varepsilon_m(3\beta_2 + 2\gamma U_0 - \gamma)]x \\ G(x) &= -\varepsilon_m\gamma U_0^2 x^2 \\ H(x) &= U_0[\tfrac{1}{2}(\varepsilon_m^2 + U_0\gamma^2)x^2 + \tfrac{3}{4}U_0\delta x] \\ I(x) &= -U_0[\varepsilon_m H^{(0)^2}(\tfrac{1}{2}\chi - 2\beta_1\beta_2) + U_0^2(\gamma^2 + \tfrac{3}{2}\delta)]x \end{aligned}\right\} \quad (63)$$

on the condition that

$$D'(x)\cos 3\Psi + E'(x)\sin 3\Psi + F'(x)\cos 2\Psi + G'(x)\sin 2\Psi + H'(x)\cos\Psi + I'(x)\sin\Psi = 0. \quad (64)$$

We could proceed to $\phi^{(3)}_x$ and thus $\phi^{(3)}$, and then to the next order to the price of more and more intricate algebraic calculations. However, the computation up to the order of $u^{(2)}$ and $\phi^{(2)}_x$ is sufficient to exhibit (i) the generation of harmonics of the source signal, (ii) the fact the $u^{(2)}$ contributes to the propagating component at the original frequency, and (iii) that the expansions obtained via the straightforward perturbation method are not uniformly valid in space since a growth like x^n is observed for the $u^{(n)}$ component, and this is physically unsound far from the source. Therefore (46) and (47) correspond to a so-called near-field solution. To obtain a solution valid far from the source (so-called far-field solution) one needs to envisage a multiple-scale technique. This is discussed in the next section. Before turning to this, some other comments are in order.

Comments on the methodology. Herein above, following previous works exemplified by the one of Thompson and Tiersten (1977), we have used a method of "variation of constants", with certain constraints imposed on these, e.g. the equation that follows (57), or eqn (64), which allow one to determine the necessary factors of trigonometric functions in the $u^{(0)}$ and $u^{(1)}$ solutions. This is also used below in Section 8. The arbitrariness of such constraints must be noted. As a matter of fact, Daher and one of the present authors (Daher and Maugin, 1989a) have recently commented upon this aspect of the "source problem" in acoustics, elasticity and piezoelectricity. In the elastic monomode case, an exact solution can be obtained by using the method of characteristics (see Maugin, 1985). For small amplitudes this exact, but implicit, solution can be expanded yielding an explicit solution of the type of (46). This expansion does not exactly coincide with the result of the direct Poincaré expansion, the spatially varying coefficients of the representations of $u^{(0)}$, $u^{(1)}$, etc. being, in general, different. Daher and Maugin (1989b) have shown† that the results could be reconciled if, instead of constraints such as (64) one imposed a continuity or continuation argument for the spatial derivative of the higher-order components $u^{(i)}$ of the displacement. The reason for this is altogether clear. The initial-boundary condition (33) is entirely accounted for by the $u^{(0)}$ solution. For higher order components, apart from a possible zero value at the source (only the fundamental should be present there), we have stated no condition. But it seems natural to assume that the components of order higher than zero

† See also Daher and Maugin (1989a). The same problem is commented upon by Cantrell *et al.* (1987).

in the stress should be zero at the source, and this we can impose in the form of the "continuation" condition for $x = 0$, $t = 0$:

$$\frac{\partial u^{(1)}}{\partial x} = \frac{\partial u^{(2)}}{\partial x} = \cdots = 0.$$

This, indeed, in the purely elastic monomode case, gives "constants" of integration in agreement with those obtained by expansion of the exact "characteristic" solution. Obviously, the discrepancy observed is not so much important for very short travelled distance and the qualitative behavior remains the same.

7. UNIFORMLY VALID FAR-FIELD SOLUTION

We now apply a multiple-scale technique for which one classically introduces a slow space variable $s = \epsilon x$ and set

$$u(x,t) = \tilde{u}(x,s,t), \quad \phi_x = \tilde{\phi}_x(x,s,t). \tag{65}$$

Then a Poincaré expansion is made for $\tilde{u}$. We note that

$$\left.\begin{array}{ll} u_t = \tilde{u}_t, & u_{tt} = \tilde{u}_{tt} \\ u_x = \tilde{u}_x + \epsilon\tilde{u}_s, & u_{xx} = \tilde{u}_{xx} + 2\epsilon\tilde{u}_{sx} + \epsilon^2\tilde{u}_{ss} \\ \phi_x = \tilde{\phi}_x, & \phi_{xx} = \tilde{\phi}_{xx} + \epsilon\tilde{\phi}_{sx} \end{array}\right\} \tag{66}$$

where $\epsilon = \epsilon_1 = \epsilon_2$. Then

$$\left.\begin{array}{l} \tilde{u} = \epsilon_1\tilde{u}^{(0)}(x,s,t) + \epsilon_1^2\tilde{u}^{(1)}(x,s,t) + \cdots \\ \tilde{\phi}_x = -\tilde{H}^o + \epsilon_2\tilde{\phi}_x^{(1)}(x,s,t) + \epsilon_2^2\tilde{\phi}_x^{(2)}(x,s,t) + \cdots. \end{array}\right\} \tag{67}$$

Substituting from eqns (66) and (67) into eqns (42) and (43) we obtain the following hierarchy of equations.

- *Order one in $\epsilon_1 = \epsilon_2$*

$$\tilde{u}_{tt}^{(0)} - \tilde{u}_{xx}^{(0)} = 0 \tag{68}$$

$$\tilde{\phi}_{xx}^{(1)} + \beta_2 H^{(0)}\tilde{u}_{xx}^{(0)} = 0. \tag{69}$$

- *Order two in $\epsilon_1 = \epsilon_2$*

$$\tilde{u}_{tt}^{(1)} - \tilde{u}_{xx}^{(1)} = 2\tilde{u}_{xs}^{(0)} + \gamma(\tilde{u}_x^{(0)^2})_x - 2\beta_1 H^{(0)}\phi_{xx}^{(1)} \tag{70}$$

and

$$\tilde{\phi}_{xx}^{(2)} + \tilde{\phi}_{xs}^{(1)} - \beta_2\tilde{\phi}_{xx}^{(1)}\tilde{u}_x^{(0)} + \beta_2 H^{(0)}\tilde{u}_{xx}^{(1)} - \beta_2\tilde{\phi}_x^{(1)}\tilde{u}_{xx}^{(0)} + 2\beta_2 H^{(0)}\tilde{u}_{sx}^{(0)} + \tfrac{1}{2}\chi H^{(0)^2}\tilde{\phi}_{xx}^{(1)} = 0. \tag{71}$$

We may consider for $u^{(0)}$ a solution in the form of a right-running plane wave as

$$\tilde{u}^{(0)} = F^{(0)}(\Psi, s), \quad \Psi = t - x \tag{72}$$

where $F^{(0)}$ is an arbitrary function and s is a parameter. The dependence on s is specified at the next step. We have

$$\tilde{u}_t^{(0)} = \partial F^{(0)}/\partial\Psi, \qquad \tilde{u}_{tt}^{(0)} = \partial^2 F^{(0)}/\partial\Psi^2$$
$$\tilde{u}_x^{(0)} = -\partial F^{(0)}/\partial\Psi, \quad \tilde{u}_{xx}^{(0)} = \partial^2 F^{(0)}/\partial\Psi^2. \tag{73}$$

Then eqn (69) yields

$$\tilde{\phi}_{xx}^{(1)} = -\beta_2 H^{(0)} \frac{\partial^2 F^{(0)}}{\partial\Psi^2}. \tag{74}$$

Substituting from eqns (72)–(74) into eqn (70) one obtains

$$\tilde{u}_{tt}^{(1)} - \tilde{u}_{xx}^{(1)} = -2\left(\frac{\partial^2 F^{(0)}}{\partial s\,\partial\Psi} + \gamma \frac{\partial F^{(0)}}{\partial\Psi}\frac{\partial^2 F^{(0)}}{\partial\Psi^2} - \varepsilon_m \frac{\partial^2 F^{(0)}}{\partial\Psi^2}\right). \tag{75}$$

We shall avoid the production of secular terms through the right-hand side of this equation by imposing that this right-hand side vanish, obtaining thus the secularity condition

$$\frac{\partial^2 F^{(0)}}{\partial s\,\partial\Psi} + \gamma \frac{\partial F^{(0)}}{\partial\Psi}\frac{\partial^2 F^{(0)}}{\partial\Psi^2} - \varepsilon_m \frac{\partial^2 F^{(0)}}{\partial\Psi^2} = 0. \tag{76}$$

If we set

$$\tilde{F} = F^{(0)} - \frac{\varepsilon_m}{\gamma}\Psi \tag{77}$$

then

$$G^{(0)} \equiv \frac{\partial\tilde{F}}{\partial\Psi} = -\frac{\partial\tilde{u}^{(0)}}{\partial x} - \frac{\varepsilon_m}{\gamma} \tag{78}$$

and we can write (76) as the simplest (first-order) nonlinear equation of wave theory as

$$\frac{\partial G^{(0)}}{\partial s} + \gamma G^{(0)}\frac{\partial G^{(0)}}{\partial\Psi} = 0. \tag{79}$$

The solution surfaces of this equation are known as (Whitham, 1974)

$$G^{(0)} = \mathscr{H}(\Psi - \gamma s G^{(0)}) \tag{80}$$

where $\mathscr{H}$ is an arbitrary function. Its expression is determined by the source condition. For an excitation such as $u(0,t) = U_0(1-\cos t)$, $t > 0$, eqn (80) takes on the form

$$G^{(0)} = -U_0 \sin(\Psi - \gamma s G^{(0)}) \tag{81}$$

or

$$-\frac{G^{(0)}}{U_0} = \sin\left[\Psi + \gamma U_0 s\left(-\frac{G^{(0)}}{U_0}\right)\right]. \tag{82}$$

Setting

$$\zeta = \Psi - \frac{s}{L_B}\frac{G^{(0)}}{U_0}, \quad L_B \equiv \frac{1}{\gamma U_0} \tag{83}$$

where one can recognize in L_B the breaking distance for shock-wave formation in nonlinear wave theory (see Nayfeh and Mook, 1979; Beyer, 1974), we rewrite (63) as

$$-(G^{(0)}/U_0) = \sin\zeta \tag{84}$$

with, for further use,

$$\mathrm{d}\Psi = \left(1 - \frac{s}{L_B}\cos\zeta\right)\mathrm{d}\zeta. \tag{85}$$

The solution (82) or (84) is implicit and, therefore, not very convenient. To find an explicit solution one considers a Fourier series expansion

$$-(G^{(0)}/U_0) = \sum_{n=1}^{\infty} B_n \sin n\Psi \tag{86}$$

with coefficients B_n given by

$$B_n = \frac{1}{\pi}\int_0^{2\pi}\left(-\frac{G^{(0)}}{U_0}\right)\sin n\Psi\,\mathrm{d}\Psi. \tag{87}$$

Using now the transformation (83)–(85) we have

$$B_n = \frac{1}{\pi}\int_0^{2\pi}\sin\zeta\left[\sin n\left(\zeta - \frac{s}{L_B}\sin\zeta\right)\left(1 - \frac{s}{L_B}\cos\zeta\right)\right]\mathrm{d}\zeta \tag{88}$$

which integrates to (Abramovitz and Stegum, 1965)

$$B_n = -\frac{2J_n(ns/L_B)}{(ns/L_B)} \tag{89}$$

where J_n is the nth Bessel function of the first kind. Thus the explicit solution for the selected source is given by

$$G^{(0)} = U_0\sum_{n=1}^{\infty}\frac{2J_n(ns/L_B)}{(ns/L_B)}\sin(n\Psi) \tag{90}$$

and thus

$$\tilde{F} = \frac{U_0}{n}\sum_{n=1}^{\infty}\frac{2J_n(ns/L_B)}{(ns/L_B)}\cos(n\Psi) + \text{const.} \tag{91}$$

and, via (72) and (77)

$$\tilde{u}^{(0)} = U_0\left[1 - \sum_{n=1}^{\infty}\frac{2J_n(ns/L_B)}{(n^2 s/L_B)}\cos(n\Psi) - \varepsilon_m(x/L_B)\right]. \tag{92}$$

The solution (90) or (92) is of the same type as the well known solution of Fubini-Ghiron (1935) in nonlinear acoustics. It is built of harmonics with pseudo-periodic coefficients. The displacement solution (92) in addition involves a term proportional to the phase or the x

coordinate. This corresponds to a spatially uniform strain† due to the presence of the bias magnetic field and magnetomechanical couplings. If necessary, this can be translated in terms of a constant internal stress as is the case in the magnetostriction of uniformly magnetized bodies (see Maugin, 1979b).

8. SPECIAL CASE: CLASSICAL MAGNETOSTRICTION

We next proceed to linearize eqns (1) and (2a) in terms of the elastic displacement but *not* of the magnetic field. In this case we obtain the following two equations which we shall refer to as those of classical magnetostriction (compare the fully linearized case in Section 4):

$$\rho_R \ddot{u}_i = \delta_{Ri}(C_{KRMN}u_{M,N} + B'''_{KRMN}\phi_{,M}\phi_{,N})_{,K} \tag{93}$$

$$\mu_{KL}\phi_{,LK} - B''_{KLMN}(\phi_{,L}u_{M,N})_{,K} = 0. \tag{94}$$

By projection onto the orthonormal basis $\{l^{\alpha}\}$ of Section 3 we have

$$\rho_R u^{\alpha}_{tt} = m_{\alpha}u^{\alpha}_{XX} + B_{\alpha}(\phi^2_X)_X \tag{95}$$

$$\bar{\mu}\phi_{XX} - \sum_{\beta}\bar{B}''_{\beta}(\phi_X u^{\beta}_X)_X = 0 \tag{96}$$

where we have set

$$\left.\begin{aligned} m_{\alpha} &= C_{KRMN}\lambda_N\lambda_K l^{\alpha}_M l^{\alpha}_R = l^{\alpha}_M\Gamma_{MR}l^{\alpha}_R \\ \Gamma_{MR} &= C_{KRMN}\lambda_N\lambda_K \\ B_{\alpha} &= B_{KRMN}\lambda_M\lambda_N\lambda_K l^{\alpha}_R \\ \bar{B}''_{\beta} &= B''_{KLMN}\lambda_K\lambda_N\lambda_L l^{\beta}_N \\ \bar{\mu} &= \mu_{KL}\lambda_K\lambda_L. \end{aligned}\right\} \tag{97}$$

Having now recourse to the monomode hypothesis, introducing dimensionless quantities, dropping the index α and asterisks and using x instead of X, we have the following nondimensional system of two partial differential equations [compare the system (42) and (43)]:

$$u_{tt} - u_{xx} = \beta_1(\phi^2_x)_x \tag{98}$$

$$\phi_{xx} - \beta_2(\phi_x u_x)_x = 0 \tag{99}$$

where

$$\beta_1 = \frac{B\tilde{H}^2}{m} = 0(10^{-5}), \quad \beta_2 = \frac{\bar{B}_0}{m} = 0(1). \tag{100}$$

Substitution of ϕ_{xx} given by eqn (99) into eqn (98) yields

$$u_{tt} - u_{xx} = \beta_1\beta_2[(\phi_x)^2u_{xx} + 2(\phi_x)^2_x u_x]. \tag{101}$$

In the sequel of this section we treat eqns (101) and (99) by the same methods as in the fully nonlinear case.

† Spatially uniform strains can also appear in this type of solution for purely elastic bodies, depending on the type of initial-boundary conditions. For these, see Cantrell *et al.* (1987).

8.1. *Solution by means of a straightforward expansion*

We consider expansions [where the ϵs are defined in eqns (48)]

$$\left.\begin{aligned} u(x,t) &= \epsilon_1 u^{(0)} + \epsilon_1^2 u^{(1)} + \epsilon_1^3 u^{(2)} + \cdots \\ \phi_x(x,t) &= -H^{(0)} + \epsilon_2 \phi_x^{(1)} + \epsilon_2^2 \phi_x^{(2)} + \cdots \end{aligned}\right\} \tag{102}$$

and this results in the following hierarchy of coupled systems.

- *Order one in ϵ_1 and ϵ_2 (with $\epsilon_1 = 0(\epsilon_2)$)*

$$u_{tt}^{(0)} - u_{xx}^{(0)} = 0 \tag{103}$$

$$\phi_{xx}^{(1)} + \beta_2 H^{(0)} u_{xx}^{(0)} = 0 \tag{104}$$

with direct solutions in the form

$$u^{(0)} = U_0(1 - \cos \Psi), \quad \Psi = t - x \tag{105}$$

$$\phi_{xx}^{(1)} = -\beta_2 H^{(0)} u_{xx}^{(0)} = -\beta_2 H^{(0)} U_0 \cos \Psi \tag{106}$$

for a source condition $u(x = 0, t) = U_0(1 - \cos t)$. By integration one finds

$$\phi_x^{(1)} = \beta_2 U_0 H^{(0)} \sin \Psi.$$

- *Order two in ϵ_1 and ϵ_2*

$$u_{tt}^{(1)} - u_{xx}^{(1)} = -2\varepsilon_m u_{xx}^{(0)} \tag{107}$$

$$\phi_{xx}^{(2)} - \beta_2[-H^{(0)} u_{xx}^{(1)} + (\phi_x^{(1)} u_x^{(0)})_x] = 0, \tag{108}$$

a system which has the solution

$$u^{(1)} = A(x) \sin \Psi + B(x) \cos \Psi + C(x) \tag{109}$$

with

$$\left.\begin{aligned} A(x) &= 2\varepsilon_m U_0 x \\ B(x) &= 0 \\ C(x) &= -2\varepsilon_m U_0 \end{aligned}\right\} \tag{110}$$

under the condition that

$$C'(x) + A'(x) \sin \Psi + B'(x) \cos \Psi = 0.$$

Obviously, from (108) and (106) one also obtains

$$\phi_{xx}^{(2)} = \beta_2^2 H^{(0)} U_0[-2\beta_1 H^{(0)^2}(x \sin \Psi + \cos \Psi) + U_0 \sin 2\Psi] \tag{111}$$

and this integrates to

$$\phi_x^{(2)} = \beta_2^2 H^{(0)} U_0 \left[-2\beta_1 H^{(0)^2} x \cos \Psi + \frac{U_0}{2}(\cos 2\Psi - 1) \right].$$

- *Order three in ϵ_1 and ϵ_2*

$$u^{(2)}_{tt} - u^{(2)}_{xx} = -2\beta_1\beta_2[H^{(0)2}u^{(1)}_{xx} - 2H^{(0)}\phi^{(1)}_x u^{(0)}_{xx} - H^{(0)}\phi^{(1)}_{xx}u^{(0)}_x] \tag{112}$$

and

$$\phi^{(3)}_{xx} - \beta_2[-H^{(0)}u^{(2)}_{xx} + \phi^{(1)}_x u^{(1)}_{xx} + \phi^{(2)}_x u^{(0)}_{xx} + \phi^{(1)}_{xx}u^{(1)}_x + \phi^{(2)}_{xx}u^{(0)}_x] = 0. \tag{113}$$

After some calculation eqn (112) becomes

$$u^{(2)}_{tt} - u^{(2)}_{xx} = \varepsilon_m\beta_2U_0[-4\beta_1H^{(0)2}(x\sin\Psi+\cos\Psi)+3U_0\sin 2\Psi]. \tag{114}$$

By integration this yields

$$u^{(2)} = D(x)\sin 2\Psi + E(x)\cos 2\Psi + F(x)\sin\Psi + G(x)\cos\Psi \tag{115}$$

with

$$\left.\begin{aligned} D(x) &= 0 \\ F(x) &= -4\varepsilon_m^2U_0x \\ G(x) &= 2\varepsilon_m^2U_0x^2 \\ E(x) &= -\tfrac{3}{2}\varepsilon_m\beta_2U_0^2x. \end{aligned}\right\} \tag{116}$$

This global solution up to order three in ϵ_1 and ϵ_2 exhibits the generation of harmonics due *uniquely* to magnetoelastic couplings of the magnetostrictive type (this is certainly not very efficient) but the solutions obtained are not uniformly valid along the spatial axis.

8.2. *Solution by means of the multiple-scale technique*

We set ($\epsilon = \epsilon_1 = \epsilon_2$)

$$s = \epsilon x, \quad u(x,t) = \tilde{u}(x,s,t), \quad \phi_x(x,t) = \tilde{\phi}_x(x,s,t) \tag{117}$$

so that eqns (46) and (47) hold true, and then we consider the expansions (67). Now we obtain the following hierarchy of systems.

- *Order one in ϵ_1 and ϵ_2*

$$\tilde{u}^{(0)}_{tt} - \tilde{u}^{(0)}_{xx} = 0 \tag{118}$$

$$\phi^{(1)}_{xx} + \beta_2H^{(0)}\tilde{u}^{(0)}_{xx} = 0. \tag{119}$$

- *Order two in ϵ_1 and ϵ_2*

$$u^{(1)}_{tt} - u^{(1)}_{xx} = 2\tilde{u}^{(0)}_{sx} + 2\varepsilon_m\tilde{u}^{(0)}_{xx} \tag{120}$$

$$\tilde{\phi}^{(2)}_{xx} + \tilde{\phi}^{(1)}_{sx} - \beta_2[(-H^{(0)}\tilde{u}^{(1)}_{xx} + \tilde{\phi}^{(1)}_x\tilde{u}^{(0)}_{xx}) + (-2H^{(0)}\tilde{u}^{(0)}_{sx} + \tilde{\phi}^{(1)}_{xx}\tilde{u}^{(0)}_x)] = 0 \tag{121}$$

where eqn (119) has been used to transform eqn (120). We need not consider higher orders.

The solution of eqn (118) as a right-running wave again is

$$\tilde{u}^{(0)} = F^{(0)}(\Psi, s), \quad \Psi = t - x. \tag{122}$$

We proceed as in Section 7 but now the secularity condition (76) is replaced by

$$\frac{\partial^2 F^{(0)}}{\partial \Psi \, \partial s} - \varepsilon_m \frac{\partial^2 F^{(0)}}{\partial \Psi^2} = 0 \tag{123}$$

or, setting

$$G^{(0)} = \frac{\partial F^{(0)}}{\partial \Psi} \tag{124}$$

we have the equation

$$\frac{\partial G^{(0)}}{\partial s} - \varepsilon_m \frac{\partial G^{(0)}}{\partial \Psi} = 0 \tag{125}$$

which is a linear first-order partial differential equation which integrates immediately by the method of characteristics to

$$G^{(0)} = \mathscr{H}(\Psi + \varepsilon_m s) \tag{126}$$

where the function $\mathscr{H}$ will be specified by the source condition. For a source (44) we have thus

$$G^{(0)} = U_0 \sin (\Psi + \varepsilon_m s). \tag{127}$$

By integration this produces the displacement solution

$$\begin{aligned} \bar{u}^{(0)} = F^{(0)} &= U_0[1 - \cos (\Psi + \varepsilon_m s) \\ &= U_0\{1 - \cos [t - x(1 - \varepsilon_m)]\}. \end{aligned} \tag{128}$$

9. REMARK ON EQUATIONS (42) AND (43)

If we keep the complete equations (42) and (43) with nonlinearities of all origins and use the straightforward expansion method in terms of the small parameters ϵ_1 and ϵ_2 at order one in these parameters we shall obtain the same system as (49) and (50) while for higher order we shall have the following.

- *Order two in ϵ_1 and ϵ_2*

$$u_{tt}^{(1)} - u_{xx}^{(1)} = \gamma(u_x^{(0)^2})_x - \beta_1 H^{(0)}(2\phi_{xx}^{(1)} - H^{(0)} u_{xx}^{(0)}) \tag{129}$$

$$\phi_{xx}^{(2)} - \beta_2[(\phi_x^{(1)} u_x^{(0)})_x - H^{(0)} u_{xx}^{(1)}] + \tfrac{1}{2}\chi H^{(0)^2} \phi_{xx}^{(1)} + \lambda H^{(0)}(u_x^{(0)^2})_x = 0. \tag{130}$$

- *Order three in ϵ_1 and ϵ_2*

$$\begin{aligned} u_{tt}^{(2)} - u_{xx}^{(2)} = {} & 2\gamma(u_x^{(0)} u_x^{(1)})_x + \delta(u_x^{(0)^3})_x - 2\beta_1 H^{(0)} \phi_{xx}^{(2)} \\ & + \beta_1(\phi_x^{(1)^2})_x - 2\beta_1 H^{(0)}(\phi_x^{(1)} u_x^{(0)})_x + \beta_1 H^{(0)} u_{xx}^{(1)}, \end{aligned} \tag{131}$$

$$\begin{aligned} & \phi_{xx}^{(3)} - \beta_2[(\phi_x^{(1)} u_x^{(1)})_x + (\phi_x^{(2)} u_x^{(0)})_x - H^{(0)} u_{xx}^{(2)}] \\ & + \tfrac{1}{2}\chi[H^{(0)^2} \phi_{xx}^{(2)} - H^{(0)}(\phi_x^{(1)^2})_x] - \lambda[(\phi_x^{(1)} u_x^{(0)^2})_x - H^{(0)}(u_x^{(0)} u_x^{(1)})_x] = 0. \end{aligned} \tag{132}$$

This case is no more difficult to deal with than the somewhat simplified case of Section 5. In particular we obtain

$$\left.\begin{aligned} u^{(0)} &= U_0(1-\cos\Psi), \quad \Psi = t-x \\ \phi^{(1)}_{,x} &= -\beta_2 H^{(0)} U_0 \cos\Psi \\ \phi^{(1)}_{,t} &= \beta_2 H^{(0)} U_0 \sin\Psi \end{aligned}\right\} \tag{133}$$

and

$$\left.\begin{aligned} u^{(1)} &= A(x)\cos 2\Psi + B(x)\sin\Psi + C(x) \\ A(x) &= \frac{\gamma U_0^2}{2}x \\ B(x) &= \beta_1 U_0 H^{(0)^2}(1+2\beta_2)x \\ C(x) &= -\frac{\gamma U_0^2}{2}x - \beta_1 U_0 H^{(0)^2}(1+2\beta_2) \end{aligned}\right\} \tag{134}$$

while

$$\begin{aligned} \phi^{(2)}_{,x} = H^{(0)} U_0 \beta_2 \Bigg\{ & \left[U_0(\beta_2-\gamma) + \frac{\lambda}{\beta_2}\right]\sin 2\Psi + 2\gamma U_0 x\cos 2\Psi \\ & -\beta_1 H^{(0)^2}(1+2\beta_2)x\sin\Psi - \left[\beta_1(1+2\beta_2) - \frac{\chi}{2}\right] H^{(0)^2}\cos\Psi \Bigg\} \end{aligned} \tag{135}$$

which, by integration, yields

$$\begin{aligned} \phi^{(2)}_{,t} = -H^{(0)} U_0 \beta_2 \Bigg\{ & \gamma U_0 x \sin 2\Psi - \beta_1 H^{(0)^2}(1+2\beta_2)x\cos\Psi \\ & -\frac{1}{2}\left(U_0\beta_2 + \frac{\lambda}{\beta_2}\right)\cos 2\Psi - \tfrac{1}{2}\chi H^{(0)^2}\sin\Psi + \frac{1}{2}\left(U_0\beta_2 + \frac{\lambda}{\beta_2}\right)\Bigg\}. \end{aligned} \tag{136}$$

A comparison between these results and the results (58) and (59) reveals that the only difference is in additional terms which only alter the value of some numerical coefficients [e.g. β_2 is replaced by $\beta_2+1/2$, $(\beta_2-\gamma)$ replaced by $(\beta_2-\gamma)+(\lambda/\beta_2 U_0)$, $\chi/2$ replaced by $(\chi/2)-\beta_1(1+2\beta_2)$].

10. NONLINEAR VIBRATIONS OF RESONATORS

We now consider a medium of finite extent (length $2h$) in one of its dimensions, plane waves travelling back and forth between these two limiting surfaces (Fig. 1). This may be

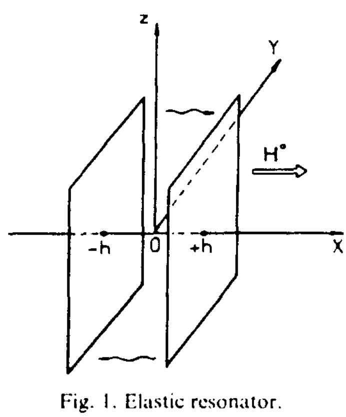

Fig. 1. Elastic resonator.

called a resonator if the vibration is adapted to the thickness $2h$. We want to examine the effects of nonlinearities on this resonance phenomenon when the end surfaces $x = \pm h$ are free of stresses. We shall have to solve both internal (for $|x| < h$) and external ($|x| > h$) problems. For $|x| > h$ we have a vacuum for which Maxwell's magnetostatic equations reduce to

$$\phi_{xx} = 0. \tag{137}$$

For $|x| < h$ we have eqns (39) and (40) with the last terms neglected (for a one-dimensional monomode motion):

$$u_{tt} - u_{xx}(1 + 2\gamma u_x + 3\delta u_x^2) = \beta_1(\phi_x^2)_x \tag{138}$$

$$\phi_{xx} - \beta_2(\phi_x u_x)_x + \tfrac{1}{6}\chi(\phi_x^3)_x = 0, \tag{139}$$

the estimates (41) holding true. We also need the boundary conditions at $|x| = h$ in the same nondimensionalized form. We have $N_K = (\pm 1, 0, 0)$ at $x = \pm h$ and it is not difficult to show that the conditions (6b) and (6c) take on the following form:

$$[\![-\phi_x + \beta_2(\phi_x u_x) - \tfrac{1}{6}\chi\phi_x^3]\!] = 0 \tag{140}$$

$$[\![\phi]\!] = 0. \tag{141}$$

10.1. *Treatment by means of the straightforward expansion*

The travelled distance $2h$ is certainly small as compared to the characteristic distance L_H introduced in previous sections. Accordingly, we need not bother with questions of validity over long spatial intervals and it is quite sufficient to address the problem of solving simultaneously eqns (137) through (141) by using the fact that the parameters ϵ_1 and ϵ_2 defined in eqns (48) are small and considering naïve, straightforward expansions in these small parameters assumed to be of the same order. We set thus

$$\begin{aligned} u(x,t) &= \epsilon_1 u^{(0)}(x,t) + \epsilon_1^2 u^{(1)}(x,t) + \epsilon_1^3 u^{(2)}(x,t) + \cdots \\ \phi_x(x,t) &= -H^{(0)} + \epsilon_2 \phi_x^{(1)}(x,t) + \epsilon_2^2 \phi_x^{(2)}(x,t) + \cdots. \end{aligned} \tag{142}$$

Substituting from (142) into eqns (138)–(141) we obtain the following hierarchy of boundary-value problems (in fact, matching with an external solution insofar as ϕ is concerned).

- *Order one in ϵ_1 and ϵ_2*

$$\left.\begin{aligned} u_{tt}^{(0)} - u_{xx}^{(0)} &= 0 \quad (143) \\ \phi_{xx}^{(1)} + \beta_2 H^{(0)} u_{xx}^{(0)} &= 0 \quad (144) \end{aligned}\right\} \text{ for } |x| < h$$

$$\left.\begin{aligned} u_x^{(0)} &= 0 \quad (145) \\ [\![\phi_x^{(1)} + \beta_2 H^{(0)} u_x^{(0)} - \tfrac{1}{6}\chi H^{(0)3}]\!] &= 0 \quad (146) \end{aligned}\right\} \text{ at } |x| = h.$$

- *Order two in ϵ_1 and ϵ_2*

$$u_{tt}^{(1)} - u_{xx}^{(1)} = \gamma(u_x^{(0)})_x^2 - 2\beta_1 H^{(0)} \phi_{xx}^{(1)} \tag{147}$$

$$\phi_{xx}^{(2)} - \beta_2[(\phi_x^{(1)} u_x^{(0)})_x - H^{(0)} u_{xx}^{(1)}] + \tfrac{1}{2}\chi H^{(0)2} \phi_{xx}^{(1)} = 0 \quad \text{for } |x| < h \tag{148}$$

$$\left.\begin{aligned} u_x^{(1)} &= 0 \\ [\![\phi_x^{(2)} - \beta_2(-H^{(0)}u_x^{(1)} + \phi_x^{(1)}u_x^{(0)}) + \tfrac{1}{2}\chi H^{(0)^2}\phi_x^{(1)}]\!] &= 0 \end{aligned}\right\} \quad \text{at } |x| = h. \tag{149, 150}$$

● *Order three in ϵ_1 and ϵ_2*

$$u_{tt}^{(2)} - u_{xx}^{(2)} = 2\gamma(u_x^{(0)}u_x^{(1)})_x + \delta(u_x^{(0)})_x^3 + \beta_1[(\phi_x^{(1)^2})_x - 2H^{(0)}\phi_{xx}^{(2)}] \tag{151}$$

$$\phi_{xx}^{(3)} - \beta_2[(\phi_x^{(1)}u_x^{(1)})_x + (\phi_x^{(2)}u_x^{(0)})_x - H^{(0)}u_{xx}^{(2)}] + \tfrac{1}{2}\chi[H^{(0)^2}\phi_{xx}^{(2)} - H^{(0)}(\phi_x^{(1)^2})_x] = 0 \quad \text{for } |x| < h \tag{152}$$

$$u_x^{(2)} = 0 \tag{153}$$

$$[\![\phi_x^{(3)} - \beta_2(-H^{(0)}u_x^{(2)} + \phi_x^{(1)}u_x^{(1)} + \phi_x^{(2)}u_x^{(0)}) + \tfrac{1}{6}\chi H^{(0)}(-3\phi_x^{(1)^2} + 2H^{(0)}\phi_x^{(2)})]\!] = 0 \quad \text{at } |x| = h. \tag{154}$$

The solution of (143) obviously is a superposition of an incident wave with wave number $K = 1$ and a reflected wave with wave number $K = -1$, hence

$$u^{(0)} = \frac{U_0}{2}[\cos(t-x) - \cos(t+x)],$$

i.e.

$$U^{(0)} = (U_0 \sin t)\sin x. \tag{155}$$

This is a standing wave. The boundary condition (145) imposes that

$$h = (2q+1)\frac{\pi}{2} = N\frac{\pi}{2} = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \ldots \tag{156}$$

where q is an integer and $N = 2q+1$ is called the partial mode of the elastic resonator. Also, from (144), we get

$$\phi_{xx}^{(1)} = -\beta_2 H^{(0)}u_{xx}^{(0)} = \beta_2 H^{(0)}U_0 \sin t \sin x. \tag{157}$$

We look for the field solution $H^{(1)} = -\phi_x^{(1)}$ for $|x| < h$ on account of the fact that $\phi_{xx}^{(1)}$ satisfies (157) inside the slab, $|x| < h$, eqn (137) outside the slab and the jump conditions (146) across the interfaces. Outside the slab

$$\phi_{xx}^{(1)\text{ext}} = 0 \quad \text{for } |x| > h \tag{158}$$

with

$$\phi \to 0 \quad \text{as } |x| \to \infty. \tag{159}$$

Equation (158) yields $\phi_x^{(1)\text{ext}} = C_3$ and $\phi^{(1)\text{ext}} = C_3x + C_4$, but both constants C_3 and C_4 must vanish by virtue of (159). Thus

$$\phi_x^{(1)\text{ext}} = 0. \tag{160}$$

We recall that

$$[\![\phi_x^{(0)}]\!] = -[\![H^{(0)}]\!] = 0. \tag{161}$$

Also, from (157)

$$\phi_{xx}^{(1)\mathrm{int}} = -\beta_2 H^{(0)} U_0 \sin t \sin x \tag{162}$$

which by integration gives

$$\phi_x^{(1)\mathrm{int}} = \beta_2 H^{(0)} U_0 \sin t \cos x + C_1(t) \tag{163}$$

where C_1 can be determined from the condition (146) which, on account of (138), (160) and (161), can be rewritten as

$$\phi_x^{(1)\mathrm{int}}(x = \pm h) = 0. \tag{164}$$

Consequently

$$C_1 = -\beta_2 H^{(0)} U_0 \sin t \cos h \tag{165}$$

and

$$\phi_x^{(1)} = 2\beta_2 H^{(0)} U_0 \sin t \cos\left(\frac{x+h}{2}\right) \cos\left(\frac{x-h}{2}\right) \quad \text{for } |x| < h. \tag{166}$$

Now we can proceed to the $u^{(1)}$ solution.

10.2. *Anisochronism*

Upon substituting (157) into eqn (147) we obtain

$$u_{tt}^{(1)} - u_{xx}^{(1)} = -\tfrac{1}{2}\gamma U_0^2 \sin 2x(1-\cos 2t) - 2\varepsilon_m U_0 \sin t \sin x. \tag{167}$$

This is to be integrated between $-h$ and $+h$ on account of the limit conditions (149). The solution of eqn (167) obviously is the sum of a particular solution and the general solution of the homogeneous equation. Therefore, we assume a solution in the form

$$u^{(1)}(x,t) = A(x) \cos 2t + B(x) \sin 2t + C(x) \sin t + D(x). \tag{168}$$

Substituting in eqn (167) we obtain a set of problems for ordinary differential equations for A, B, C and D as

$$\left.\begin{array}{ll} A''(x) + 4A(x) = -\tfrac{1}{2}\gamma U_0^2 \sin 2x, & |x| < h \quad (169a) \\ A'(x) = 0 & \text{at } |x| = h \quad (169b) \end{array}\right\}$$

$$\left.\begin{array}{ll} B''(x) + 4B(x) = 0, & |x| < h \quad (170a) \\ B'(x) = 0 & \text{at } |x| = h \quad (170b) \end{array}\right\}$$

$$\left.\begin{array}{ll} C''(x) + C(x) = 2\varepsilon_m \sin x, & |x| < h \quad (171a) \\ C'(x) = 0 & \text{at } |x| = h \quad (171b) \end{array}\right\}$$

$$\left.\begin{array}{ll} D''(x) = \tfrac{1}{2}\gamma U_0 \sin 2x, & |x| < h \quad (172a) \\ D'(x) = 0 & \text{at } |x| = h \quad (172b) \end{array}\right\}.$$

The solution of (169a) satisfying the imposed boundary condition (169b) is obtained as

$$A(x) = -\frac{\gamma U_0^2}{16}(\sin 2x - x\cos 2x) \tag{173}$$

while (170a, b) is easily shown to give

$$B(x) = 0. \tag{174}$$

Then eqn (172a), on account of (172b), integrates to

$$D(x) = -\frac{\gamma U_0^2}{8}(\sin 2x + 2x). \tag{175}$$

It is clear that the term in $C(x)$ in (168) contributes at the fundamental frequency (here one) so that the integral of eqns (171a, b) provide the most interesting term for the effect called anisochronism. The solution of eqn (171a) is built of a particular solution, say $C^P(x)$, of the complete equation, and a general solution, say $C^G(x)$, of the homogeneous equation. The latter we take as $C^G(x) = U_0 \sin \Lambda x$, where Λ is a perturbed (about one) wave number defined by

$$\Lambda = 1 + \delta_A. \tag{176}$$

If $\delta_A = 0$, the solution $C^G(x)$ of eqn (171a) satisfies the linear approximation. If $\delta_A \neq 0$, then eqn (171a) is approximately satisfied and, substituting for the total solution $C(x) = C^P(x) + C^G(x)$ in the imposed boundary condition (171b), we obtain that δ_A satisfies the following condition:

$$U_0(1+\delta_A)\cos(1+\delta_A)h + \varepsilon_m U_0 h \sin h - \varepsilon_m U_0 \cos h = 0. \tag{177}$$

But $C'(x) = 0$ at $|x| = h = \pi/2$ for the partial mode of the first order, and since δ_A is assumed to be small we can use the approximation $\cos(1+\delta_A)h \simeq -\delta_A h$ which, inserted in eqn (177), delivers δ_A as

$$\delta_A = \varepsilon_m. \tag{178}$$

This small quantity is the alteration in the fundamental mode of vibrations of the resonator resulting from the nonlinear magnetoelastic properties of the body. It varies like the square of the bias magnetic field and is directly proportional to the magnetoacoustic coupling coefficient, eqn (32), which usually causes a reduction in the speed of elastic waves in magnetostrictive materials. This effect is also obvious in the solution (128). For an anisochronism due to nonlinear elastic properties only, one must proceed to the elastic solution up to the order of $u^{(2)}$ (compare Planat, 1984; Maugin, 1985). A more involved study of the nonlinear vibrations of magnetostrictive elastic resonators shall be given later on (Abd-Alla and Maugin, 1988).

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